ALMOST CONFORMAL TRANSFORMATION IN A FOUR DIMENSIONAL RIEMANNIAN MANIFOLD WITH AN ADDITIONAL STRUCTURE

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Abstract. We consider a 4-dimensional Riemannian manifold M with a metric g and affinor structure q. The local coordinates of these tensors are circulant matrices. Their first orders are (A, B, C, B), $A, B, C \in FM$ and $(0, 1, 0, 0)$, respectively.

We construct another metric $\tilde{g}$ on M. We find the conditions for $\tilde{g}$ to be a positively defined metric, and for q to be a parallel structure with respect to the Riemannian connection of g.

Further, let x be an arbitrary vector in T_pM, where p is a point on M. Let φ and ϕ be the angles between x and qx, x and q^2x with respect to g. We express the angles between x and qx, x and q^2x with respect to $\tilde{g}$ with the help of the angles φ and ϕ.

Also,we construct two series $\{\varphi_n\}$ and $\{\phi_n\}$. We prove that every of it is an increasing one and it is converge.

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1. Introduction

The main purpose of the present paper is to continue the investigations in [1], [2], [3]. We study a class of Riemannian manifolds which admits a circulant metric g and an additional circulant structure q. The forth degree of structure q is an identity, and q is a parallel structure with respect to the Riemannian connection ∇ of g.

2. Preliminaries

We consider a 4-dimensional Riemannian manifold M with a metric g and an affinor structure q. We note the local coordinates of g and q are circulant matrices. The next conditions and results have been discussed in [3].

The metric g have the coordinates:

$$g_{ij} = \begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}, \qquad A > C > B > 0 \tag{1}$$

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in the local coordinate system (x_1, x_2, x_3, x_4), and $A = A(p), B = B(p), C = C(p)$, where $p(x_1, x_2, x_3, x_4) \in F \subset R^4$. Naturally, A, B, C are smooth functions of a point p. We calculate that $det g_{ij} = (A-C)^2((A+C)^2 - 4B^2) \neq 0$.

Further, let the local coordinates of q be

$$q_i^{\cdot j} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

We will use the notation $\Phi_i = \dfrac{\partial \Phi}{\partial x^i}$ for every smooth function Φ defined in F.

We know from [3] that the following identities are true

$$q^4 = E; \quad q^2 \neq \pm E; \tag{3}$$

$$g(qw, qv) = g(w, v), \quad w, v \in \chi M, \tag{4}$$

where E is the unit matrix;

$$0 < B < C < A \quad \Rightarrow \quad g \ is \ possitively \ defined. \tag{5}$$

Now, let $w = (x, y, z, u)$ be a vector in χM. Using (1) and (2) we calculate that

$$g(w, w) = A(x^2 + y^2 + z^2 + u^2) + 2B(xy + xu + yz + zu) + 2C(xz + yu) \tag{6}$$

$$g(w, qw) = (A + C)(xu + xy + yz + zu) + B(x^2 + y^2 + z^2 + u^2 + 2xz + 2yu) \tag{7}$$

$$g(w, q^2w) = 2A(xz + yu) + 2B(xu + xy + zy + zu) + C(x^2 + y^2 + z^2 + u^2). \tag{8}$$

Let M be the Riemannian manifold with a metric g and an affinor structure q, defined by (1) and (2), respectively. Let $w(x, y, z, u)$ be no eigenvector on T_pM(i.e. $w(x, y, z, u) \neq (x, x, x, x)$, $w(x, y, z, u) \neq (x, -x, x, -x)$). If φ is the angle between x and qx, and ϕ is the angle between x and q^2x, then we have $\cos\varphi = \dfrac{g(w, qw)}{g(w, w)}$, $\cos\phi = \dfrac{g(w, q^2w)}{g(w, w)}$, $\varphi \in (0, \pi)$, $\phi \in (0, \pi)$.

We apply (6), (7) and (8) in the above equations and we get

$$\cos\varphi = \frac{(A + C)(xu + xy + yz + zu) + B(x^2 + y^2 + z^2 + u^2 + 2xz + 2yu)}{A(x^2 + y^2 + z^2 + u^2) + 2B(xy + xu + yz + zu) + 2C(xz + yu))}, \tag{9}$$

$$\cos\phi = \frac{C(x^2 + y^2 + z^2 + u^2) + 2B(xy + xu + yz + zu) + 2A(xz + yu)}{A(x^2 + y^2 + z^2 + u^2) + 2B(xy + xu + yz + zu) + 2C(xz + yu))}. \tag{10}$$

3. Almost conformal transformation in M

Let M satisfies (1)– (5). We note $f_{ij} = g_{ik} q_t^k q_j^t$, i.e.

$$f_{ij} = \begin{pmatrix} C & B & A & B \\ B & C & B & A \\ A & B & C & B \\ B & A & B & C \end{pmatrix}. \tag{11}$$

We calculate $det f_{ij} = (C - A)^2((A + C)^2 - 4B^2) \neq 0$, so we accept f_{ij} for local coordinates of another metric f. The metric f_{ij} is necessarily undefined. Further,

we suppose α and β are two smooth functions in $F \subset R^4$ and we construct the metric $\tilde{g}$, as follows:

$$\tilde{g} = \alpha.g + \beta.f. \tag{12}$$

We say that equation (12) define an almost conformal transformation, noting that if $\beta = 0$ then (12) implies the case of the classical conformal transformation in M [2].

From (1), (2), (11) and (12) we get the local coordinates of $\tilde{g}$:

$$\tilde{g}_{ij} = \begin{pmatrix} \alpha A + \beta C & (\alpha+\beta)B & \alpha C + \beta A & (\alpha+\beta)B \\ (\alpha+\beta)B & \alpha A + \beta C & (\alpha+\beta)B & \alpha C + \beta A \\ \alpha C + \beta A & (\alpha+\beta)B & \alpha A + \beta C & (\alpha+\beta)B \\ (\alpha+\beta)B & \alpha C + \beta A & (\alpha+\beta)B & \alpha A + \beta C \end{pmatrix}. \tag{13}$$

We see that f_{ij} and $\tilde{g}_{ij}$ are both circulant matrices.

Theorem 3.1. [3] *Let M be a Riemannian manifold with a metric g from (1) and an affinor structure q from (2). Let ∇ be the Riemannian connection of g. Then $\nabla q = 0$ if and only if, when*

$$gradA = (gradC)q^2; \quad 2gradB = (gradC)(q + q^3). \tag{14}$$

Theorem 3.2. *Let M be a Riemannian manifold with a metric g from (1) and an affinor structure q from (2). Also, let $\tilde{g}$ be a metric of M, defined by (12). Let ∇ and $\tilde{\nabla}$ be the corresponding connections of g and $\tilde{g}$, and $\nabla q = 0$. Then $\tilde{\nabla} q = 0$ if and only if, when*

$$grad\alpha = grad\beta.q^2; \quad grad\beta = -grad\beta.q^2. \tag{15}$$

Proof. At first we suppose (15) is valid. Using (15) and (14) we can verify that the following identity is true:

$$\text{grad}(\alpha A+\beta C) = \text{grad}(\alpha C+\beta A).q^2, \ 2\text{grad}(\alpha+\beta)B = \text{grad}(\alpha C+\beta A).(q+q^3) \tag{16}$$

The identity (16) is analogue to (14), and consequently we conclude $\tilde{\nabla} q = 0$.

Inversely, if $\tilde{\nabla} q = 0$ then analogously to (14) we have (16). Now, (14) and (16) imply the system

$$A\text{grad}\alpha + C\text{grad}\beta = (C\text{grad}\alpha + A\text{grad}\beta)q^2 \tag{17}$$

$$2B(\text{grad}\alpha + \text{grad}\beta) = (C\text{grad}\alpha + A\text{grad}\beta)(q + q^3). \tag{18}$$

From (17) we find the only solution $\text{grad}\alpha = \text{grad}\beta.q^2$, and from (18) we get the only solution $\text{grad}\beta = -\text{grad}\beta.q^2$. So the theorem is proved. □

Lemma 3.3. *Let $\tilde{g}$ be the metric given by (12). If $0 < \beta < \alpha$ and g is positively defined, then $\tilde{g}$ is also positively defined.*

Proof. From the condition $(\alpha - \beta)(A - C) > 0$ we get $\alpha A + \beta C > \beta A + \alpha C > 0$. Also, we see that $\beta A + \alpha C > (\alpha + \beta)B > 0$ and finely $(\alpha A + \beta C) > \beta A + \alpha C > (\alpha + \beta)B > 0$. Analogously to (5) we state that $\tilde{g}$ is positively defined. □

Lemma 3.4. *Let $w = w(x(p), y(p), z(p), u(p))$ be in T_pM, $p \in M$, $qw \neq w$, $q^2w \neq w$ and g and $\tilde{g}$ be the metrics of M, related by (12). Then we have:*

$$\tilde{g}(w, w) = (\alpha A + \beta C)(x^2 + y^2 + z^2 + u^2) + 2(\alpha + \beta)B(xy + xu + yz + zu)+ \\ 2(\alpha C + \beta A)(yu + xz)$$

$$\tilde{g}(w,qw) = (\alpha+\beta)(A+C)(xu+xy+yz+zu)+ \qquad (19)$$
$$(\alpha+\beta)B(x^2+y^2+z^2+u^2+2xz+2yu)$$

$$\tilde{g}(w,q^2w) = 2(\alpha A+\beta C)(xz+yu)+2(\alpha+\beta)B(xu+xy+zy+zu)$$
$$+(\alpha C+\beta A)(x^2+y^2+z^2+u^2).$$

Theorem 3.5. *Let $w = w(x(p), y(p), z(p), u(p))$ be a vector in T_pM, $p \in M$, $qw \neq w$, $q^2w \neq w$. Let g and $\tilde{g}$ be two positively defined metrics of M, related by (12). If φ and φ_1 are the angles between w and qw, with respect to g and $\tilde{g}$, ϕ and ϕ_1 are the angles between w and q^2w, with respect to g and $\tilde{g}$, then the following equations are true:*

$$\cos\varphi_1 = \frac{(\alpha+\beta)\cos\varphi}{\alpha+\beta\cos\phi}, \qquad (20)$$

$$\cos\phi_1 = \frac{\alpha\cos\phi+\beta}{\alpha+\beta\cos\phi}. \qquad (21)$$

Proof. Since g and $\tilde{g}$ are both positively defined metrics we can calculate $\cos\varphi$ and $\cos\varphi_1$, respectively. Then by using (13) and (19) we get (20). Also, we calculate $\cos\phi$ and $\cos\phi_1$, respectively. Then by using (13) and (19) we get (21). □

Theorem 3.5 implies immediately the assertions:

Corollary 3.6. *Let φ and φ_1 be the angles between w and qw with respect to g and $\tilde{g}$. Let ϕ and ϕ_1 be the angles between w and q^2w with respect to g and $\tilde{g}$ Then*
1)$\varphi = \frac{\pi}{2}$ if and only if when $\varphi_1 = \frac{\pi}{2}$;
2)if $\phi = \frac{\pi}{2}$ then $\phi_1 = \arccos\frac{\beta}{\alpha}$
3)if $\phi_1 = \frac{\pi}{2}$ then $\phi = \arccos(-\frac{\beta}{\alpha})$.

Further, we consider an infinite series of the metrics of M as follows:

$$g_0,\ g_1,\ g_2, \dots,\ g_n, \dots$$

where

$$g_0 = g, \quad g_1 = \tilde{g}, \quad g_n = \alpha g_{n-1} + \beta f_{n-1}, \qquad (22)$$
$$f_{n-1,is} = g_{n-1,ka} q_s^a q_i^k, \quad 0 < \beta < \alpha.$$

By the method of the mathematical induction we can see that the matrix of every g_n is circulant one and every g_n is positively defined.

Theorem 3.7. *Let M be a Riemannian manifold with metrics g_n from (22) and an affinor structure q from (2). Let $w = w(x(p), y(p), z(p))$ be in T_pM, $p \in M$, $qw \neq w$, $q^2w \neq w$. Let φ_n be the angle between w and qw, with respect to g_n, let ϕ_n be the angle between w and q^2w with respect to g_n. Then the infinite series:*

$$1) \qquad \varphi_0,\ \varphi_1,\ \varphi_2, \dots,\ \varphi_n, \dots$$

is converge and $\lim\varphi_n = 0$,

$$2) \qquad \phi_0,\ \phi_1,\ \phi_2, \dots,\ \phi_n, \dots$$

is converge and $\lim\phi_n = 0$.

Proof. Using the method of the mathematical induction and Theorem 3 we obtain:

$$\cos\varphi_n = \frac{(\alpha+\beta)\cos\varphi_{n-1}}{\alpha+\beta\cos\phi_{n-1}} \tag{23}$$

as well as $\varphi_n \in (0,\pi)$. From (23) we get:

$$\frac{\cos\varphi_n}{\cos\varphi_{n-1}} = \frac{\alpha+\beta}{\alpha+\beta\cos\varphi_{n-1}} \geq 1. \tag{24}$$

The equation (24) implies $\cos\varphi_n \geq \cos\varphi_{n-1}$, so the series $\{\cos\varphi_n\}$ is increasing one and since $\cos\varphi_n < 1$ then it is converge. From (23) we have $\lim\cos\varphi_n = 1$, so $\lim\varphi_n = 0$.

Now, we find

$$\cos\phi_n = \frac{\alpha\cos\phi_{n-1}+\beta}{\alpha+\beta\cos\phi_{n-1}} \tag{25}$$

as well as $\phi_n \in (0,\pi)$. From (25) we get:

$$\cos\phi_n - \cos\phi_{n-1} = \frac{\beta\sin^2\phi_{n-1}}{\alpha+\beta\cos\phi_{n-1}} \geq 0. \tag{26}$$

The equation (26) implies $\cos\phi_n > \cos\phi_{n-1}$, so the series $\{\cos\phi_n\}$ is increasing one and since $\cos\phi_n < 1$ then it is converge. From (25) we have $\lim\cos\phi_n = 1$, so $\lim\phi_n = 0$. $\square$

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